On axiomatizations of the Shapley value for assignment games

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1. Preliminaries
2. Negative results
3. The positive result
4. Some interesting results

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- it does not meant the axiomatization works on class of games $A \cap B$,
- it implies the axiomatization works on $A \cup B$ (!).
- Conclusion: we have to check (almost) all classes of games one by one.

Shapley value

The Shapley value (Shapley, 1953) of Player i in game v is as follows:

$$\phi_i(v) \doteq \sum_{T \subseteq N \setminus \{i\}} \frac{|T|!(|N \setminus T| - 1)!}{|N|!} v_i'(T) \ ,$$

where $v_i'(T) \doteq v(T \cup \{i\}) - v(T)$ is Player i's marginal contribution to coalition T in game v.

Axioms I

Solution ϕ on class of games $A \subseteq \mathcal{G}^N$

- is *Pareto optimal* (or *efficient*), if $\sum_{i \in N} \phi_i(v) = v(N)$ for all $v \in A$;

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- satisfies the *equal treatment property*, if $\phi_i(v) = \phi_j(v)$ for all $v \in A$ and symmetric ($v_i'(S) = v_j'(S)$, if $S \subseteq N \setminus \{i, j\}$) players i, j in v;

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- is *covariant* under strategic equivalence, if $\phi(\alpha v \oplus \beta) = \alpha \phi(v) + \beta$, for all $v \in A$, $\alpha > 0$ and $\beta \in \mathbb{R}^N$ such that $\alpha v \oplus \beta \in A$;

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- satisfies *coalitional strategic equivalence*, if $\phi_i(v) = \phi_i(v + \alpha u_T)$, for all $v \in A$, $i \in N$, $T \subseteq N \setminus \{i\}$ and $\alpha > 0$ such that $v + \alpha u_T \in A$;

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- satisfies *fairness*, if $\phi_i(v+w)-\phi_i(v)=\phi_j(v+w)-\phi_j(v)$ for all $v,w \in A$ and $i,j \in N$ such that i,j are symmetric in w and $v+w \in A$.

Further axioms

Let $A \subseteq \mathcal{G}_N = \cup_{T \subseteq N} \mathcal{G}^T$ be a class of games, ϕ be a solution on A, and for all $v \in A$, $T \subseteq N$, $T \neq \emptyset$, such that $v^{S \cup (N \setminus T)} \in A$, let

$$v_{T,\phi}(S) \doteq v(S \cup (N \setminus T)) - \sum_{i \in N \setminus T} \phi_i(v^{S \cup (N \setminus T)}) \text{ for all } S \subseteq T,\ S \neq \emptyset,$$

and $v_{T,\phi}(\emptyset) = 0$. Then $v_{T,\phi} \in \mathcal{G}^T$ is called the ϕ-*reduced game* of v on coalition T. Solution ϕ defined on $A \subseteq \mathcal{G}$

- is *HM-consistent* (Hart and Mas-Colell, 1989), briefly *consistent*, if for all $T \subseteq N$, $v \in A \cap \mathcal{G}^T$ and $S \subseteq T$, $S \neq \emptyset$, such that $v_{S,\phi} \in A$, it holds that $\phi_i(v_{S,\phi}) = \phi_i(v)$ for all $i \in S$.

Well-known results

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- Pareto optimality, the null player property and fairness (van den Brink, 2001).

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- Pareto optimality, covariance, the equal treatment property and consistency (Hart and Mas-Colell, 1989).

A trivial remark

Theorem

If $|B \cup S| = 2$, then solution ϕ on $\mathcal{G}^{B,S}$ satisfies Pareto optimality and the equal treatment property if and only if it is the Shapley value.

Let $OR(N)$ be the set of all (linear) orderings on set N. Consider the following two solutions for assignment games with buyer and seller sets B and S. First, let

$$OR_B \doteq \{\tau \in OR(B \cup S) \mid \tau(i) \leq |B| \Rightarrow i \in B\} \ ,$$

be the orders where the buyers come first, and let

$$OR_S \doteq \{\tau \in OR(B \cup S) \mid \tau(i) \leq |S| \Rightarrow i \in S\} \ ,$$

be the orders where the sellers come first.

Now, for all $v \in \mathcal{G}^{B,S}$ and $i \in B \cup S$, let

$$\phi_i^B(v) \doteq \frac{1}{|OR_B|}\Bigg(\sum_{\tau \in OR_B} (v(\{j \in B \cup S \mid \tau(j) \leq \tau(i)\}) - v(\{j \in B \cup S \mid \tau(j) < \tau(i)\}))\Bigg) \ ,$$

be the average marginal contribution of buyer or seller i over all orders where the buyers come first, and

$$\phi_i^S(v) \doteq \frac{1}{|OR_S|}\left(\sum_{\tau\in OR_S}(v(\{j\in B\cup S \mid \tau(j)\leq\tau(i)\}) - v(\{j\in B\cup S \mid \tau(j)<\tau(i)\}))\right)$$

be the average marginal contribution of buyer or seller i over all orders where the sellers come first. Then, for all $v \in \mathcal{G}^{B,S}$, let

$$\phi^{B,S}(v) \doteq \frac{\phi^B(v)+\phi^S(v)}{2}$$

be the average of these two solutions.

Results on $\phi^{B,S}(v)$

Theorem

Solution $\phi^{B,S}$ is a convex combination of random order values and satisfies anonymity, the equal treatment property, covariance, additivity, and strong monotonicity on $\mathcal{G}^{B,S}$.

An example

Consider $B=\{1,2\},\ S=\{3\}$, and $v\in\mathcal{G}^{B,S}$ determined by $a_{1,3}=1$ and $a_{2,3}=2$, that is, $v(\{1,3\})=1,\ v(\{2,3\})=v(\{1,2,3\})=2$ and $v(T)=0$ otherwise. Then $\phi^{B}(v)=(0,0,2)$, $\phi^{S}(v)=(\frac{1}{2},\frac{3}{2},0)$, and thus $\phi^{B,S}(v)=(\frac{1}{4},\frac{3}{4},1)$. However, $\phi^{Sh}(v)=(\frac{1}{6},\frac{2}{3},\frac{7}{6})$.

Negative results

Corollary

On the class $\mathcal{G}^{B,S}$ of assignment games, the following axiomatizations of the Shapley value do not work:

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- *Chun's axiomatization (Pareto optimality, the equal treatment property and coalitional strategic equivalence),*
- *van den Brink's axiomatization (Pareto optimality, the null player property and fairness).*

Potential

Definition

Let $A \subseteq \mathcal{G}_N$. For every function $P : A \to \mathbb{R}$, $T \subseteq N$, $T \neq \emptyset$, and for all $v \in \mathcal{G}^T \cap A$ and $i \in T$ such that $|T| = 1$ or $v^{T \setminus \{i\}} \in A$, let

$$P'_i(v) \doteq \begin{cases} P(v), & \text{if } |T| = 1 \\ P(v) - P(v^{T \setminus \{i\}}) & \text{otherwise.} \end{cases} \quad (1)$$

If

$$\sum_{i \in T} P'_i(v) = v(T),$$

for all $v \in \mathcal{G}^T \cap A$ such that either $|T| = 1$ or $v^{T \setminus \{i\}} \in A$ for all $i \in T$, then P is called a potential on A.

Definition

A collection $A \subseteq \mathcal{G}$ is subgame closed, if $v^{T \setminus \{i\}} \in A$ for all $T \subseteq N$ with $|T| > 1$, $i \in T$ and $v \in \mathcal{G}^T$ such that $v \in A$.

Theorem

Let $A \subseteq \mathcal{G}_N$ be a subgame closed set of games. Then function P on $\mathcal{G}$ is a potential, if and only if $P_i'(v) = \phi_i(v)$ for all $T \subseteq N$, $T \neq \emptyset$, $v \in \mathcal{G}^T \cap A$ and $i \in T$.

(semi-)Negative result on the potential

Corollary

On the class of assignment games there is a potential P such that there exists an assignment game v and a player i such that $P'_i(v) \neq \phi_i(v)$.

HM-consistency

First, let $\mathcal{G}_a$ be the collection of all assignment games, that is, $\mathcal{G}_a = \{v \in \mathcal{G}_N \mid$ there exist $B, S \subset N$, $B \neq \emptyset,\ S \neq \emptyset,\ B \cap S = \emptyset,\ B \cup S = N$ such that $v \in \mathcal{G}^{B,S}\}$.
Moreover, let solution $\overline{\phi}$ on $\mathcal{G}_a$ for each $v \in \mathcal{G}_a \cap \mathcal{G}^{B,S}$ be given by

$$\overline{\phi}_i(v) \doteq \begin{cases} \dfrac{v(B \cup S)}{|(B \cup S) \setminus NP(v)|}, & \text{if } i \notin NP(v) \\ 0 & \text{otherwise.} \end{cases}$$

It is worth noticing that for any assignment game v and player $i \notin NP(v)$, $\overline{\phi}_i(v) > 0$.

Negative result on HM-consistency

Theorem

Solution $\overline{\phi}$ satisfies Pareto optimality, anonymity, the equal treatment property, covariance and consistency on $\mathcal{G}_a$.

Submarket

Definition

Let $A \in \mathcal{A}^{B,S}$ be an assignment situation. Then $B' \cup S'$, $B' \subseteq B$, $S' \subseteq S$, $B' \cup S' \neq \emptyset$, is a submarket of A if $a_{i,j} = 0$ for all $(i,j) \in (B' \times (S \setminus S')) \cup ((B \setminus B') \times S')$.

Axioms

Definition

Solution f on $\mathcal{A}^{B,S}$ satisfies

- submarket efficiency, if for all $A \in \mathcal{A}^{B,S}$ and for all submarkets (B', S') of A, it holds that $\sum_{i \in B' \cup S'} f_i(v) = v_A(B' \cup S')$,
- valuation fairness, if for every buyer $i \in B$, every seller $j \in S$ and every pair of assignment situations $A, \overline{A} \in \mathcal{A}^{B,S}$ such that $\overline{a}_{i,j} = 0$ and $a_{g,h} = \overline{a}_{g,h}$ for all $(g,h) \in ((B \setminus \{i\}) \times S) \cup (B \times (S \setminus \{j\}))$, it holds that $f_i(A) - f_i(\overline{A}) = f_j(A) - f_j(\overline{A})$.

The positive result (á la Myerson (1977))

Theorem

The Shapley value ϕ is the unique solution for assignment situations that satisfies submarket efficiency and valuation fairness.

and a further result (compare it to Solymosi, Brugueras and Raghavan (201?))

Theorem

Consider assignment situations $A, \overline{A} \in \mathcal{A}^{B,S}$ such that for some $i \in B, j \in S$ it holds that $\overline{a}_{i,j} \geq a_{i,j}$, and $a_{g,h} = \overline{a}_{g,h}$ for all $(g,h) \in ((B \setminus \{i\}) \times S) \cup (B \times (S \setminus \{j\}))$. Then $\phi_i(\overline{A}) \geq \phi_i(A)$ and $\phi_j(\overline{A}) \geq \phi_j(A)$.

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Thank you for your attention!